

Classification and Enumeration of Minimum $(d, 3, 3)$ -Graphs for Odd d

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Consider a network of communication stations (or of computing facilities) in which certain pairs of stations are directly linked and others must communicate indirectly through a sequence of direct links. Represent the network in the usual way by means of a graph G whose nodes and edges correspond respectively to the stations and to the direct links. Reliability considerations may require, for certain integers c and v , that G should be c -connected, so that if fewer than c stations are disabled the remainder of the network will still be connected, and that each node is of valence $\geq v$, so that the tasks of a single disabled station can be distributed among v others directly linked to it. A c -connected graph in which each node is of valence $\geq v$ [resp. $=v$] is here called a $\langle c, v \rangle$ -graph [resp. (c, v) -graph].

There are various situations in which one would like to approximate a metric space M (for example, a part of the earth's surface, or of space) by the node-set of a $\langle c, v \rangle$ -graph whose number of nodes is minimum subject to certain constraints. Constraints may be imposed not only by the geometry of M and the closeness of approximation desired, but also by an upper bound β on the distance between two stations that can be directly linked. For example, if δ is the metric diameter of M and each point of M must be within distance ϵ of a station, then the (combinatorial) diameter of the graph G must be at least $(\delta - 2\epsilon)/\beta$. A $\langle c, v \rangle$ -graph of diameter $\geq d$ is called a $\langle d, c, v \rangle$ -graph and a *minimum* $\langle d, c, v \rangle$ -graph is one that has the minimum number of nodes. Similarly, a (c, v) -graph of diameter $= d$ is called a (d, c, v) -graph, and a *minimum* (d, c, v) -graph is one that has the minimum number of nodes.

In most cases of practical interest, the problem of finding a minimum $\langle c, v \rangle$ -approximation appears to be intractable, but one may attempt to find close-to-minimum approximations by experimenting with a stock T of promising trial graphs and with various embeddings of their node-sets in M . Natural candidates for inclusion in T are the minimum $\langle d, c, v \rangle$ -graphs and

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the closely related minimum (d, c, v) -graphs. This suggests the desirability of the following tasks:

- (i) evaluating $\mu\langle d, c, v \rangle$, the number of nodes of a minimum $\langle d, c, v \rangle$ -graph;
- (ii) evaluating $\mu(d, c, v)$, the number of nodes of a minimum (d, c, v) -graph;
- (iii) classifying and enumerating the minimum $\langle d, c, v \rangle$ -graphs;
- (iv) classifying and enumerating the minimum (d, c, v) -graphs.

The easiest of these is (i), which was accomplished in [2]. Task (ii) is much more difficult, even though $\mu(d, c, v) = \mu\langle d, c, v \rangle$ for many choices of (d, c, v) ; (ii) is accomplished in [4]. Task (iv) is accomplished in [3] for $(d, 1, 3)$ and $(d, 2, 3)$, and is accomplished here for $(d, 3, 3)$ when d is odd. The most quotable result is that if $\xi = (13 - (129)^{1/2})/2$, $\eta = (13 + (129)^{1/2})/2$, and

$$\zeta_j = \frac{\eta^j - \xi^j}{\eta - \xi} = 2^{1-j} \sum_{1 \leq \text{odd } i \leq j} \binom{j}{i} 13^{j-i} 129^{(i-1)/2}$$

then the number of (isomorphism types of) minimum $(d, 3, 3)$ -graphs is equal to

$$\frac{3}{4} \zeta_{2j-1} - \frac{5}{2} \zeta_{2j-2} + \frac{21}{4} \zeta_j - 22 \zeta_{j-1} + 15 \zeta_{j-2} - 4^{j-2}$$

when $d = 4j + 1 \geq 9$ and to

$$\frac{3}{4} \zeta_{2j-1} - \frac{5}{2} \zeta_{2j} + \frac{3}{4} \zeta_{j+1} - \frac{13}{4} \zeta_j + \frac{5}{2} \zeta_{j-1} - 4^{j-1} - 2^{j-1} + \frac{1}{4}$$

when $d = 4j + 3 \geq 7$.

The methods used here can be adapted to yield information about the number of minimum $(d, 3, 3)$ -graphs when d is even, but the precise determination of that number would seem to require additional ideas not included here. In any case, even values of d are ignored except in Section 1.

Alternative approaches to the minimum (d, c, v) -graphs have been provided by Myers for $(c, v) = (1, 3)$ [6], $(c, v) = (3, 3)$ [7], and $c = 1$ [8], though he solves the enumeration problem only for $(c, v) = (1, 3)$.

Two well-known books on graphical enumeration are those of Moon [5] and Harary and Palmer [1]. The first deals with labeled trees and the second is concerned mainly with various classes of unlabeled graphs. The problem solved in the present paper does not seem to be very close to anything in these books, though of course our main tools, recursion and symmetry, are common to much of enumerative analysis. It turns out that our connectivity and valence requirements force a sort of linear growth in the number of minimum $(d, 3, 3)$ -graphs, and in the rooted case it is possible to describe

this explicitly by a system of linear recursions. That is done in Section 2. By appropriate consideration of symmetries, Section 3 handles the passage from the rooted to the unrooted case.

1. SEINES AND ROOTED SEINES

In a connected graph G , two nodes x and y form a *diametral pair* if the G -distance $\delta_G(x, y)$ is equal to the diameter $\delta(G)$. Diametral pairs are unordered unless otherwise specified. A *diametral node* is one that belongs to a diametral pair. Two or more paths are *independent* if any node common to two of the paths is an end of both.

When n is even, a *simple n -seine* is a 3-valent graph of diameter $d = n + 1$ formed from three independent paths A , B , and C of length d joining the two nodes of a diametral pair, together with the $3n/2$ edges of a matching that covers all intermediate nodes of the paths. The same definition applies when n is odd, except that then two paths are of length d , one is of length $d + 1$, and there are $(3n + 1)/2$ edges in the matching. When n is odd, a *split n -seine* is a 3-valent graph of diameter $d = n + 1$ formed from three independent paths of length d joining the two nodes of a diametral pair, together with an additional node z and $(3n + 2)/2$ additional edges. (These additional edges cover z and all intermediate nodes of the paths. They form a matching except that three of them come together at z .) It is easy to verify that simple n -seines exist for all $n \geq 1$ and split n -seines exist for all odd $n \geq 1$. The numbers of (isomorphism types of) split 1-seines, simple 1-seines and simple 2-seines are respectively 1, 1, and 2. See Figures 1 and 2.

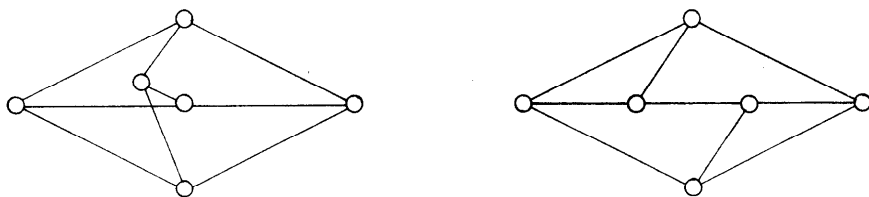


FIG. 1. The split 1-seine and the simple 1-seine.

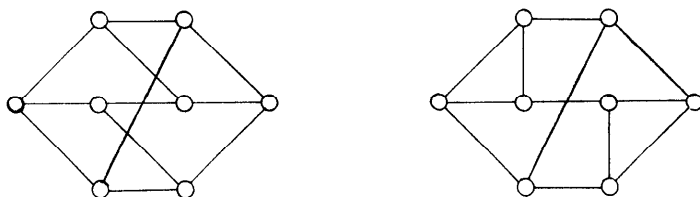


FIG. 2. The two simple 2-seines.

1.1. THEOREM. When d is odd, a graph is a minimum $(d, 3, 3)$ -graph if and only if it is a simple $(d - 1)$ -seine. When d is even, a graph is a minimum $(d, 3, 3)$ -graph if and only if it is a split $(d - 1)$ -seine or a simple $(d - 1)$ -seine.

Proof. Note first that all simple n -seines and all split n -seines are 3-connected. That is, for any two nodes p and q of such a graph S , p and q are not separated by any pair of nodes of S . To prove this, consider separately the cases in which $\{p, q\}$ intersects the diametral pair $\{x, y\}$, in which p and q are intermediate nodes of the same path A , B or C , and in which p and q are intermediate nodes of different paths. Consider also, for split n -seines, the case in which p or q is the extra node z . In each case the argument is straightforward.

To complete the proof, consider an arbitrary diametral pair $\{x, y\}$ in a 3-connected graph G of diameter d . Each path joining x and y has at least $d - 1$ intermediate nodes, and since x and y are joined by three independent paths there are at least $3d - 1$ nodes in all. If, in addition, d is even and G is 3-valent then there are at least $3d$ nodes because odd-valent graphs are of even order. But in view of the preceding paragraph, a minimum $(d, 3, 3)$ -graph has at most $3d - 1$ nodes when d is odd and at most $3d$ nodes when d is even. The desired conclusion follows readily. ■

Because of 1.1, the minimum $(d, 3, 3)$ -graphs are henceforth called $(d - 1)$ -seines. Thus all $(d - 1)$ -seines are simple when d is odd, and are simple or split when d is even. A rooted n -seine is an ordered triple (S, x, y) , where S is an n -seine and (x, y) is an ordered diametral pair of nodes of S . Two rooted n -seines (S, x, y) and (S^*, x^*, y^*) are *isomorphic* if there is a graph isomorphism of S onto S^* that carries x and y onto x^* and y^* respectively. (When S is a 1-seine or 2-seine, S has at least two diametral pairs and hence yields at least four distinct rooted seines, but all rooted seines associated with S are isomorphic.) In all that follows, the distinction between rooted seines and unrooted seines is essential. A seine not specifically described as rooted is assumed to be unrooted.

2. THE NUMBER OF ROOTED $(d - 1)$ -SEINES FOR ODD d

Henceforth there is the

STANDING HYPOTHESIS: $d = n + 1$, where n is an even integer ≥ 2 .

Let Q denote the infinite graph formed from three paths $(x, p_{11}, p_{12}, \dots)$, $(x, p_{21}, p_{22}, \dots)$ and $(x, p_{31}, p_{32}, \dots)$ which are pairwise node-disjoint except for having the same initial node x . For each positive integer k let Q_k denote the subgraph of Q spanned by $\{x\} \cup (\bigcup_1^k P_i)$, where $P_i = \{p_{1i}, p_{2i}, p_{3i}\}$. A k -start is a graph T formed by adding to Q_k the edges of a matching M such that

- (a) each member of M joins two points of $\bigcup_1^k P_i$ not joined in Q_k ,
- (b) each point of $\bigcup_1^{k-1} P_i$ is covered by M , and
- (c) $\delta_T(x, p_{1k}) = \delta_T(x, p_{2k}) = \delta_T(x, p_{3k}) = k$, where δ is the usual graph-theoretic distance.

A consequence of (a) and (c) is that if $\{p_{li}, p_{mj}\} \in M$ then $l \neq m$ and $|i - j| \leq 1$. This and the following fact are used frequently without explicit reference.

2.1. PROPOSITION. *Every automorphism of a k -start T carries x onto x and carries P_i onto P_i for $1 \leq i \leq k$.*

Proof. Note that P_k is the set of all nodes of T -valence ≤ 2 , and that for $1 \leq i < k$, P_i is the set of all nodes at T -distance $k - i$ from P_k . ■

When T is a k -start, a node $p \in P_k$ is *matched* or *unmatched* in T according as its T -valence is 2 or 1. The *end set* of T is the set of all $r \in \{1, 2, 3\}$ such that p_{kr} is matched in T . The cardinality $\rho(T)$ of T 's end set is called the *reach* of T . The *end group* of T is the group of all permutations π of $\{1, 2, 3\}$ such that T admits an automorphism (a graph isomorphism of T onto T) carrying p_{kr} onto $p_{k\pi(r)}$ for $r \in \{1, 2, 3\}$. The cardinality $\sigma(T)$ of T 's end group is called the *style* of T . Plainly $\sigma(T) \in \{1, 2, 3, 6\}$, for the end group is a subgroup of the symmetric group S_3 . Figure 3 shows all isomorphism types of k -starts for $k \leq 3$. Under each example the pair $\rho(T)$, $\sigma(T)$ is shown.

For each pair of integers $r \in \{0, 1, 2, 3\}$ and $s \in \{1, 2, 3, 6\}$, let $g_{rs}(k)$ denote the number of isomorphism types of k -starts of reach r and style s . Note that the number of isomorphism types of rooted n -seines is

$$(g_{31} + g_{32} + g_{33} + g_{36})(n).$$

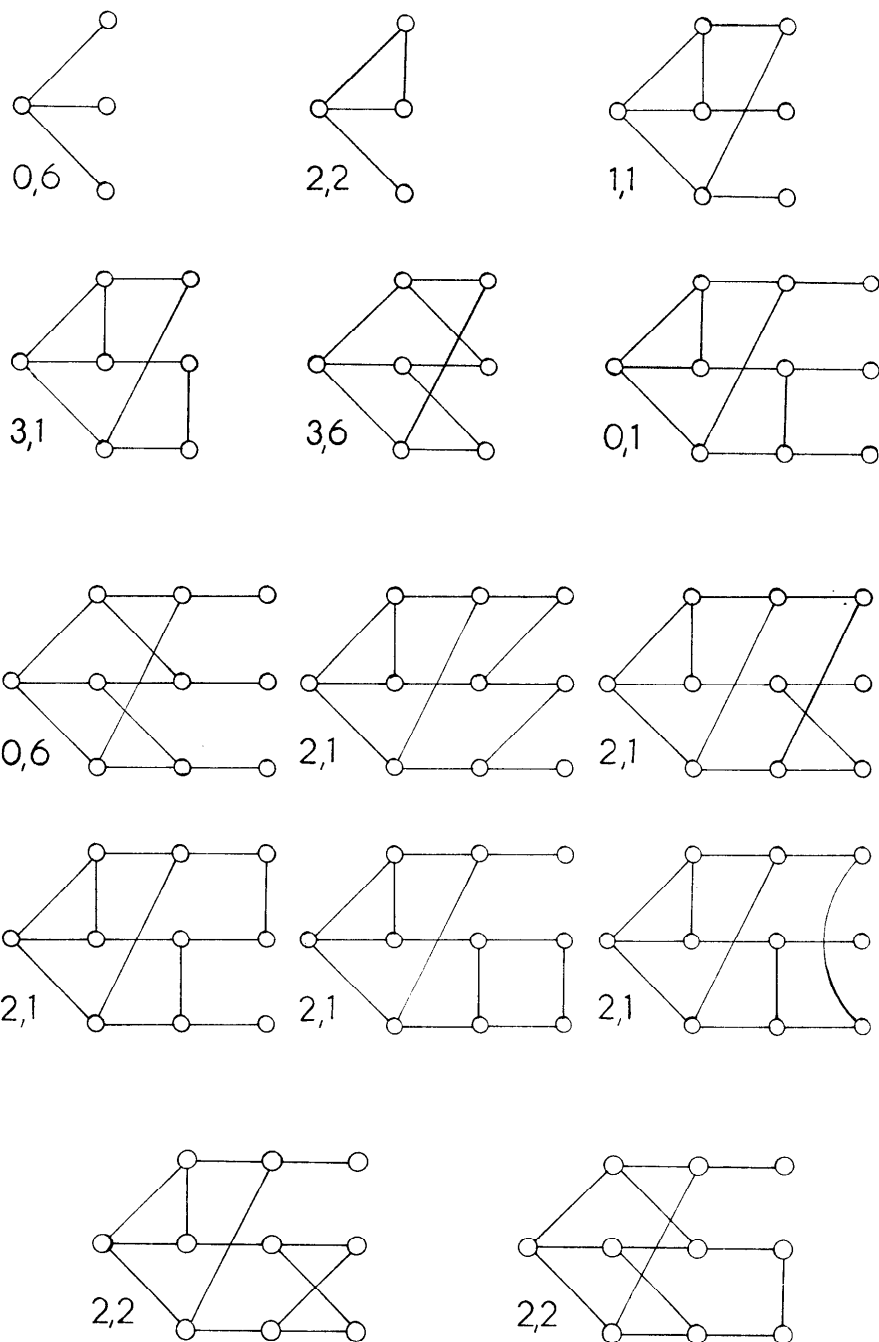
We now proceed by recursion to determine the various functions g_{rs} .

2.2. PROPOSITION. *Of the sixteen functions g_{rs} for $r \in \{0, 1, 2, 3\}$ and $s \in \{1, 2, 3, 6\}$, all but g_{01} , g_{06} , g_{11} , g_{21} , g_{22} , g_{31} and g_{36} are identically zero.*

Proof. To see that $g_{23} = g_{26} = 0$, note that if $\{1, 2\}$ is the end set of a k -start T then each automorphism of T is the identity on P_k or interchanges p_{k1} and p_{k2} .

That $g_{12} = g_{13} = g_{16} = 0$ follows from valency and distance considerations. Suppose, for example, that $\{1\}$ is the end set of a k -start T and $\{p_{(k-1)2}, p_{k1}\}$ is an edge of T . Then the T -valence of p_{k1} is 2 while that of p_{k2} and p_{k3} is 1. Also, the T -distance from p_{k1} is 2 for p_{k2} and >2 for p_{k3} . Hence each automorphism of T is the identity on P_k .

That $g_{02} = g_{03} = g_{32} = g_{33} = 0$ follows from an induction that is based on the initial conditions


 FIG. 3. Examples of all isomorphism type of k -starts for $k \leq 3$.

(a) $g_{06}(1) = g_{22}(1) = 1$, $g_{rs}(1) = 0$ for all $(r, s) \notin \{(0, 6), (2, 2)\}$ in conjunction with the observation that for all $k \geq 1$,

(b) $g_{0s}(k+1) = g_{3s}(k)$ for all s , and

(c) $g_{3s}(k+1) = g_{0s}(k)$ for all $s \neq 1$.

To establish (b), use 2.1 to verify that if a k -start of reach 3 and a $(k+1)$ -start of reach 0 are associated with the same matching M then they have the same end group. For (c), consider an arbitrary $(k+1)$ -start T of reach 3 and style $s \neq 1$, and let ϕ be an automorphism of T that is not the identity on P_{k+1} . If $\{p_{(k+1)1}, p_{(k+1)2}\}$ is an edge of T then ϕ carries $p_{(k+1)3}$ onto itself and interchanges $p_{(k+1)1}$ and $p_{(k+1)2}$, whence ϕ interchanges p_{k1} and p_{k2} . A contradiction then arises from the fact that precisely one of p_{k1} and p_{k2} is joined to $p_{(k+1)3}$. It follows that no edge of T joins two points of P_{k+1} , whence T is formed from a k -start T' of reach 0 by adding, for each $p \in P_k$, a node q_p that is not joined to p but is joined to both members of $P_k \sim \{p\}$. Then T and T' have the same end group, and (c) follows. ■

The order of the g_{rs} 's in the next result is chosen so as to simplify a later computation.

2.3. PROPOSITION. *For each positive integer k let the column vector z_k be the transpose of*

$$(g_{06}(k), g_{22}(k), g_{01}(k), g_{11}(k), g_{31}(k), g_{36}(k), g_{21}(k)).$$

Then $z_{k+1} = Az_k$ where A is the matrix

$$\begin{bmatrix} . & . & . & . & . & 1 & . \\ . & . & . & 1 & . & 1 & . \\ . & . & . & . & 1 & . & . \\ . & 1 & . & . & . & . & 2 \\ . & 1 & 2 & . & . & . & 2 \\ 1 & . & . & . & . & . & . \\ . & . & . & 2 & 3 & . & . \end{bmatrix}$$

Proof. The first and third rows are justified by (b) above, the sixth by (c). For the fourth row, note that each $(k+1)$ -start T of reach 1 is also of style 1 by 2.2, and T has as subgraph a k -start of reach 2. Each k -start of reach 2 and style 1 <resp. 2> yields 2 <resp. 1> isomorphism types of $(k+1)$ -starts of reach 1.

To justify the fifth row of A , note that $g_{11}(k+1)$ <resp. $2g_{01}(k)$ > is the number of isomorphism types of $(k+1)$ -starts of reach 3 and style 1 in which there is <resp. is not> an edge joining two nodes of P_{k+1} . Then use the fact, provided by A 's fourth row, that $g_{11}(k+1) = 2g_{21}(k) + g_{22}(k)$.

For the second and seventh rows of A , consider an arbitrary $(k+1)$ -start T of reach 2. If two nodes of P_{k+1} are joined in T then T has as subgraph a k -start T' of reach 3 and it follows with the aid of 2.2 that the style of T' is 1 (resp. 6) when that of T is 1 (resp. 2). Further, each k -start of reach 3 and style 1 (resp. 6) yields 3 (resp. 1) isomorphism types of $(k+1)$ -starts of reach 2 and style 1 (resp. 2). If no edge of T joins two nodes of P_{k+1} then T has as subgraph a k -start T' of reach 1 whose style is also 1 by 2.1. Each such T' yields 2 (resp. 1) isomorphism types of $(k+1)$ -starts of reach 2 and style 1 (resp. 2). ■

2.4. THEOREM. Let $\xi = (13 - (129)^{1/2})/2$, $\eta = (13 + (129)^{1/2})/2$, and for each positive integer j let

$$\zeta_j = \frac{\eta^j - \xi^j}{\eta - \xi} = 2^{1-j} \sum_{1 \leq \text{odd } i \leq j} \binom{j}{i} 13^{j-i} 129^{(i-1)/2}$$

Then the functions g_{rs} that are not identically 0 are as tabulated below.

	$k = 2j$	$k = 2j + 1$
$g_{01}(k)$	0	$\frac{1}{2}(-1 + 3\zeta_j) - 5\zeta_{j-1}$
$g_{06}(k)$	0	1
$g_{11}(k)$	$\frac{1}{2}(1 + 3\zeta_j) - 8\zeta_{j-1} + 10\zeta_{j-2}$	0
$g_{21}(k)$	0	$\frac{1}{2}(-1 + 15\zeta_j) - 31\zeta_{j-1} + 20\zeta_{j-2}$
$g_{22}(k)$	0	$\frac{3}{2}(1 + \zeta_{j+1}) - 23\zeta_j + 72\zeta_{j-1} - 40\zeta_{j-2}$
$g_{31}(k)$	$\frac{1}{2}(-1 + 3\zeta_j) - 5\zeta_{j-1}$	0
$g_{36}(k)$	1	0

Proof. Let A be the matrix of 2.3. A first inspection shows that 0 is an eigenvalue of A with associated eigenvector $(0, 2, 0, 0, 0, 0, -1)^t$. When $\lambda \neq 0$ the matrix $A - \lambda I$ is row-equivalent to the matrix

$$\begin{bmatrix} -\lambda & . & . & . & . & 1 & . \\ . & -\lambda & . & 1 & . & 1 & . \\ . & . & -\lambda & . & 1 & . & . \\ . & . & . & 1 - \lambda^2 & . & 1 & 2\lambda \\ . & . & . & 1 & 2 - \lambda^2 & 1 & 2\lambda \\ . & . & . & . & . & 1 - \lambda^2 & . \\ . & . & . & 2 & 3 & . & -\lambda \end{bmatrix}.$$

Hence 1 and -1 are eigenvalues of A , and associated eigenvectors turn out to be $(2, 3, -1, 1, -1, 2, -1)^t$ and $(2, 3, -1, -1, 1, -2, -1)^t$ respectively.

When $\lambda^2 \notin \{0, 1, 2\}$ the matrix M is row-equivalent to the upper triangular matrix whose first four rows are those of M and whose lower right 3×3 minor is

$$\begin{bmatrix} (1 - \lambda^2)(2 - \lambda^2) & -\lambda^2 & -2\lambda^3 \\ . & 1 - \lambda^2 & . \\ . & . & \lambda^4 - 13\lambda^2 + 10 \end{bmatrix}$$

It follows that when $\lambda \in \{-\xi^{1/2}, \xi^{1/2}, -\eta^{1/2}, \eta^{1/2}\}$, λ is an eigenvalue of A with associated eigenvector

$$(0, \psi_{\lambda^2}, \lambda^2, \lambda^3 - 2\lambda, \lambda^3, 0, 5\lambda^2 - 4),$$

where

$$\psi_{\tau} = \tau^2 - 12\tau + 8.$$

There is now enough information to diagonalize A by means of a similarity transformation, but it is computationally more convenient to work with A^2 instead.

The eigenvalues of A^2 are 0, 1, ξ and η , and for all but the first of these an independent pair of eigenvectors is available. Replacing each of these pairs with half of its sum and half of its difference, thereby obtaining other independent eigenvectors for the eigenvalue in question, and arranging the seven eigenvectors in an order that is convenient for subsequent computation, yields the matrix

$$B = \begin{bmatrix} 2 & . & . & . & . & . & . \\ 3 & 2 & \psi_{\xi} & . & . & . & \psi_{\eta} \\ -1 & . & \xi & . & . & . & \eta \\ . & . & . & \xi^{3/2} - 2\xi^{1/2} & \eta^{3/2} - 2\eta^{1/2} & 1 & . \\ . & . & . & \xi^{3/2} & \eta^{3/2} & -1 & . \\ . & . & . & . & . & 2 & . \\ -1 & -1 & 5\xi - 4 & . & . & . & 5\eta - 4 \end{bmatrix}.$$

The columns of B are eigenvectors of A^2 associated with the respective eigenvalues 1, 0, ξ , ξ , η , 1, η . By a standard result on diagonalization of matrices,

$$A^2 = BDB^{-1}$$

where

$$D = \text{diag}(1, 0, \xi, \xi, \eta, 1, \eta).$$

For each integer $j > 0$,

$$z_{2j+1} = (A^2)^j z_1 = BD^j B^{-1} z_1.$$

Let $x = B^{-1}z_1$, so that $Bx = z_1$. To compute x , recall that $z_1 = (1, 1, 0, 0, 0, 0, 0)^t$ and use row operations to transform the matrix (B, z_1) into the following:

$$\left[\begin{array}{cccccccc} 1 & . & . & . & . & . & . & \frac{1}{2} \\ . & 1 & \frac{1}{2}\psi_\xi & . & . & . & \frac{1}{2}\psi_\eta & -\frac{1}{4} \\ . & . & \xi & . & . & . & \eta & \frac{1}{2} \\ . & . & . & \xi^{1/2} & \eta^{1/2} & . & . & . \\ . & . & . & . & \eta^{1/2}(\eta - \xi) & . & . & . \\ . & . & . & . & . & 1 & . & . \\ . & . & . & . & . & . & \eta(\eta - \xi) & (3 - \xi)/2 \end{array} \right].$$

Back substitution then shows that

$$x = \left(\frac{1}{2}, \mu, \frac{\eta - 3}{2\xi(\eta - \xi)}, 0, 0, 0, \frac{\xi - 3}{2\eta(\xi - \eta)} \right)^t$$

where the value of the second entry μ is immaterial for our purpose. It follows that

$$D^j x = \frac{1}{2} \left(1, 0, \xi^{j-1} \frac{\eta - 3}{\eta - \xi}, 0, 0, 0, \eta^{j-1} \frac{\xi - 3}{\xi - \eta} \right),$$

whence the successive entries of $z_{2j+1} = BD^j x$ are

$$g_{06}(2j + 1) = 1,$$

$$g_{22}(2j + 1) = \frac{3}{2} + \frac{1}{2(\eta - \xi)} [\psi_\xi \xi^{j-1}(\eta - 3) - \psi_\eta \eta^{j-1}(\xi - 3)]$$

$$g_{01}(2j + 1) = -\frac{1}{2} + \frac{1}{2(\eta - \xi)} [\xi^j(\eta - 3) - \eta^j(\xi - 3)],$$

$$g_{11}(2j + 1) = g_{31}(2j + 1) = g_{36}(2j + 1) = 0,$$

and

$$\begin{aligned} g_{21}(2j + 1) = & -\frac{1}{2} + \frac{1}{2(\eta - \xi)} \\ & \times [(5\xi - 4) \xi^{j-1}(\eta - 3) - (5\eta - 4) \eta^{j-1}(\xi - 3)]. \end{aligned}$$

By simple manipulation and use of the fact that $\xi\eta = 10$, the formulas for $g_{r3}(2j + 1)$ stated in 2.4 are derived from the ones just obtained.

Note, for example, that

$$\begin{aligned}
 & \frac{1}{\eta - \xi} [\psi_{\xi} \xi^{j-1} (\eta - 3) - \psi_{\eta} \eta^{j-1} (\xi - 3)] \\
 &= \frac{3}{\eta - \xi} [(\eta^2 - 12\eta + 8) \eta^{j-1} - (\xi^2 - 12\xi + 8) \xi^{j-1}] \\
 &\quad - \frac{\xi \eta}{\eta - \xi} [(\eta^2 - 12\eta + 8) \eta^{j-2} - (\xi^2 - 12\xi + 8) \xi^{j-2}] \\
 &= 3(\zeta_{j+1} - 12\zeta_j + 8\zeta_{j-1}) - 10(\zeta_j - 12\zeta_{j-1} + 8\zeta_{j-2}) \\
 &= 3\zeta_{j+1} - 46\zeta_j + 144\zeta_{j-1} - 80\zeta_{j-2}.
 \end{aligned}$$

The formulas for $g_{r3}(2j)$ follow from the formulas for $g_{r3}(2j-1)$ and the fact that $z_{2j} = Az_{2j-1}$. ■

2.5. THEOREM. For even $n \geq 2$ the number of isomorphism types of rooted n -seines is $\frac{1}{2}(-1 + 3\zeta_{n/2}) - 5\zeta_{(n-2)/2}$.

Proof. Since $g_{32} = g_{33} = 0$, the number in question is $g_{31}(n) + g_{30}(n)$. Use 2.4. ■

By 2.5 there are 14 types of rooted 4-seines. The 8 types (S, x, y) shown in Figure 4 are *reversible*, meaning that (S, x, y) is isomorphic with (S, y, x) . The 3 types shown in Figure 5 are irreversible. The remaining 3 irreversible types are of the form (S, y, x) where (S, x, y) is as in Figure 5.

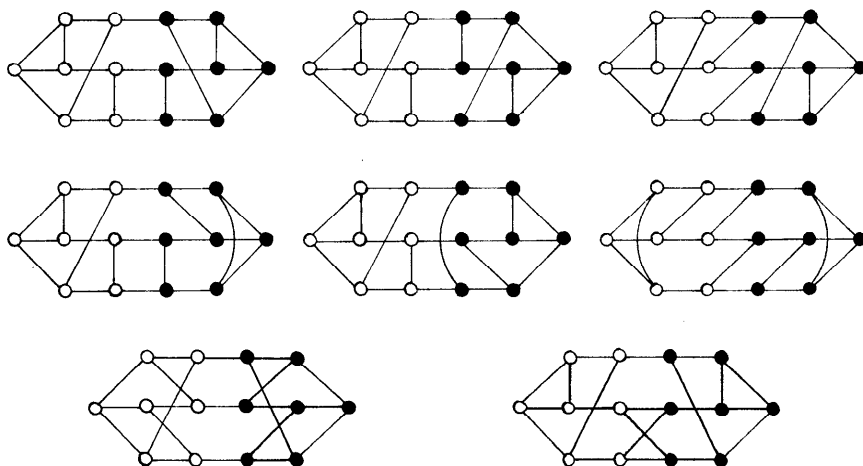
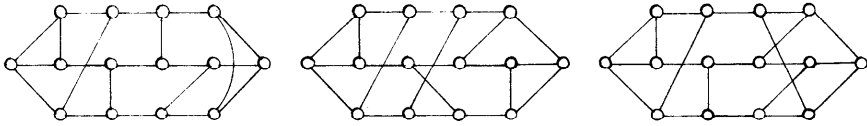


FIG. 4. The eight types of reversible rooted 4-seine (S, x, y) .


 FIG. 5. Three of the six types of irreversible rooted 4-seine (S, x, y) .

A *link* is any of the five graphs shown in Figure 6. In each link L , the six nodes are partitioned into a set L_w of three white nodes and a set L_b of three black nodes. When G is a link and p a node of G , or G is a k -start and $p \in P_k$, the node p is *matched* if its G -valence is 2; otherwise the G -valence of p is 1 and p is *unmatched*. Each reversible rooted n -seine can be formed, for some r between 0 and 3, by amalgamating a link L having r unmatched white nodes with two copies, T_w and T_b , of an $(n/2)$ -start of reach r . In the amalgamation, L 's unmatched white (resp. black) nodes are identified with the matched nodes of T_w (resp. T_b) and L 's matched white (resp. black) nodes are identified with the unmatched nodes of T_w (resp. T_b). In Figure 4, five of the examples use the link L_5 ; the last examples on the successive rows use L_3 , L_3 and L_2 respectively.

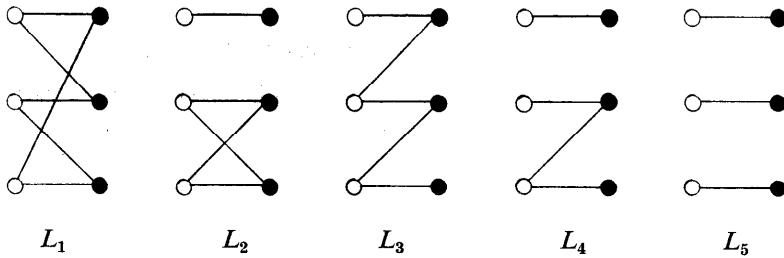


FIG. 6. The five links.

2.6. THEOREM. For even $n \geq 4$ the number of isomorphism types of reversible rooted n -seines is

$$\frac{1}{2}(1 + 21\zeta_j) - 44\zeta_{j-1} + 30\zeta_{j-2} \quad \text{when} \quad n = 4j$$

and

$$1 + \frac{1}{2}(3\zeta_{j+1} - 13\zeta_j) + 5\zeta_{j-1} \quad \text{when} \quad n = 4j + 2.$$

Proof. For each link L having r unmatched white nodes, and for each $(n/2)$ -start T of reach r and specified style, the number of isomorphism types of reversible rooted n -seines that can be formed by amalgamating L with two copies of T is shown in the table of Figure 7.

Link number	Reach of start	Style of start	Number of types
1	0	1	1
1	0	6	1
2	1	1	1
3	1	1	2
4	2	1	2
4	2	2	1
5	3	1	4
5	3	6	1

FIG. 7. Number of types of reversible rooted n -seines yielded by certain amalgamations.

Justification of the table is facilitated by a more formal description of the amalgamation process for forming reversible rooted n -seines. For each link L , let $A(L)$ denote the set of all automorphisms α of L such that $\alpha L_w = L_b$. For each k -start T and each $\alpha \in A(L)$, let $W(T, \alpha)$ denote the set of all one-to-one mappings $\omega: P_k \rightarrow L_w$ such that

- (a) ω carries matched $\langle \text{resp. unmatched} \rangle$ nodes of P_k onto unmatched $\langle \text{resp. matched} \rangle$ nodes of L_w , and
- (b) the mapping $\omega^{-1}\alpha^2\omega: P_k \rightarrow P_k$ belongs to the end group of T .

Note that if $W(T, \alpha) \neq \emptyset$ then by (a) the reach of T is equal to the number of unmatched white nodes of L . For T, L, α and ω as described, with $2k = n$, the reversible rooted $(2k)$ -seine $R = R(T, L, \alpha, \omega)$ is constructed in the manner set forth in the next paragraph.

Let ϕ be an isomorphism of T onto a graph T' that is node-disjoint from both T and L . Let μ be an automorphism of T whose restriction to P_k is $\omega^{-1}\alpha^2\omega$. Let $x' = \phi(x)$, $P'_k = \phi P_k$, and define the mapping $\beta: P'_k \rightarrow L_b$ by $\beta = \alpha\omega\mu^{-1}\phi^{-1}$. The graphs T, L and T' are amalgamated to form an n -seine S by identifying, as an individual node of S , each of the pairs $\{p, \omega(p)\}$ for $p \in P_k$ and each of the pairs $\{p', \beta(p')\}$ for $p' \in P'_k$. To see that the rooted n -seine $R = (S, x, x')$ is reversible, let $\psi(q) = \phi(\mu(q))$ for each node q of T and $\psi(q') = \phi^{-1}(q')$ for each node q' of T' . Plainly ψ is an automorphism of $T \cup T'$ that interchanges x and x' . By means of the identifications, ψ is defined also on the nodes of L . On L_w , $\psi = \beta\phi\mu\omega^{-1}$, which is equal to α because $\beta = \alpha\omega\mu^{-1}\phi^{-1}$ by definition. And since $\omega\mu\omega^{-1} = \alpha^2$, on L_b it is true that

$$\psi = \omega\phi^{-1}\beta^{-1} = \omega\phi^{-1}\phi\mu\omega^{-1}\alpha^{-1} = \omega\mu\omega^{-1}\alpha^{-1} = \alpha^2\alpha^{-1} = \alpha.$$

Thus ψ is an automorphism of S and an isomorphism of (S, x, x') onto (S, x', x) . A straightforward argument shows that every reversible rooted n -seine can be constructed in this way.

In justifying the rows of the table in Figure 7, let the nodes of the k -start T that belong to P_k be denoted by 1, 2 and 3. It may be assumed without loss of generality that the matched members of P_k have smaller indices than the unmatched members, and then because of the symmetries of L it may be assumed that the identification of nodes of P_k with those of L_w is as shown in Figure 8.

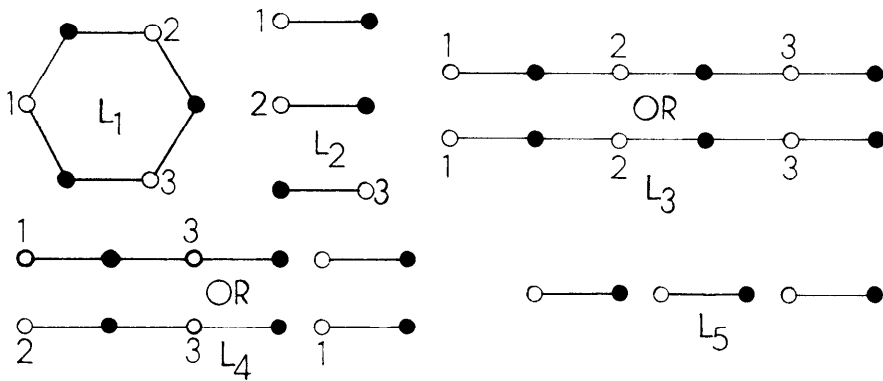


FIG. 8. Identification of nodes of p_k with white nodes of link L_i .

Now recall that α is an automorphism of L for which $L_w = L_b$, whence α interchanges the sets L_w and L_b . Consider first the case in which T is of style 1, whence α^2 is the identity. In the case of L_1 this fixes α and hence (for a given ω and ϕ) also fixes β , so that only one type of $R(T, L, \alpha, \omega)$ emerges. In the case of L_2 there are two possibilities for α and hence for β , but the two pairs of identifications (ω, β) are equivalent under the automorphism group of L_2 and thus only one type of R emerges. In the case of L_3 or L_4 , α is fixed and hence β is fixed but (as shown in Figure 8) there are two essentially different possibilities for ω ; thus in each case two types of R emerge. In the case of L_5 the four possibilities for α lead to the four possibilities for (ω, β) indicated in Figure 9 and hence to four different types of R .

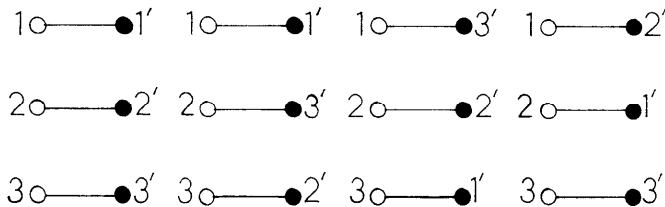


FIG. 9. Different identification pairs (ω, β) when L is L_5 and T is of style 1.

Only the second, sixth and eighth rows of the table remain. The number of possibilities for α is 3, 1 and 6 respectively. However, because of the nature of T 's end groups, in each case only a single type of R emerges.

From the table it follows that the total number of reversible rooted n -seines is

$$(g_{01} + g_{06} + 3g_{11} + 2g_{21} + g_{22} + 4g_{31} + g_{36})(n/2),$$

whence the stated formulas follow with the aid of 2.4. ■

3. THE NUMBER OF UNROOTED $(d - 1)$ -SEINES FOR ODD d

This section also comes under the

STANDING HYPOTHESIS: $d = n + 1$, where n is an even integer ≥ 2 .

Each diametral pair $\{x, y\}$ of nodes of an n -seine S is joined by at least one set of three independent paths of length d , and for a given $\{x, y\}$ there may be more than one such set of paths. It is often convenient to pick a particular set $A = (x, a_1, \dots, a_n, y)$, $B = (x, b_1, \dots, b_n, y)$, $C = (x, c_1, \dots, c_n, y)$ and carry on the discussion with respect to it. Note that, in the matching involved in the definition of a simple n -seine, a_i is matched with one of the six points $b_{i-1}, b_i, b_{i+1}, c_{i-1}, c_i, c_{i+1}$, there being two exclusions when i is 1 or n . This fact is used frequently without explicit reference.

Each diametral node x of an n -seine belongs to a unique diametral pair $\{x, y\}$, but there may be diametral pairs disjoint from $\{x, y\}$. An n -seine with more than one diametral pair is called *ambiguous*. The first 2-seine in Figure 2 has four diametral pairs. It is in that respect an anomaly, as the following result shows.

3.1. THEOREM. *If S is an ambiguous n -seine for even $n \geq 4$, there are cliques $\{x, x', w\}$ and $\{y, y', z\}$ in S such that*

$$\delta_S(x, y) = \delta_S(x', y') = n + 1, \quad \delta_S(w, z) = n - 1,$$

and $\{x, y\}$ and $\{x', y'\}$ are the only diametral pairs of S .

Proof. Let $\{x, y\}$ and $\{x', y'\}$ be distinct diametral pairs of S and let $A = (x, a_1, \dots, a_n, y)$, $B = (x, b_1, \dots, b_n, y)$ and $C = (x, c_1, \dots, c_n, y)$ be three independent paths from x to y . Since $\delta_S(x', y') = d$ and each pair of the paths forms a circuit of length $2d$, it may be assumed without loss of generality that $x' = a_j$ and $y' = c_k$ with $j + k = d$ and $1 \leq j \leq k \leq h$. If a_j is matched with c_r the sequence $(a_j, c_r, c_{r+1}, \dots, c_k)$ is a path of length $< d$, and the situation is similar if c_k is matched with a_s .

Hence a_j is matched with b_r and c_k with b_s , where

$$|r - s| + 2 \geq d.$$

It follows that $r = 1$ and $s = n$, whence j is 1 or 2.

If $j = 2$, $\{a_2, b_1\}$ is an edge, whence a_1 is not matched with b_2 for if it were there would be no admissible mate for c_1 . Thus a_1 is matched with c_t for $t \in \{1, 2\}$, whence (with $n \geq 4$) $t < d - 2$ and $(a_2, a_1, c_t, \dots, c_{d-2})$ is a path of length $< d$ from x' to y' . Since this is contradictory, it follows that $j = 1$. That is, $x' = a_1$, $y' = c_n$, and $\{a_1, b_1\}, \{c_n, b_n\}$ are edges of S . With $w = b_1$ and $z = b_n$, $\{x, x', w\}$ and $\{y, y', z\}$ are the desired cliques. Note also, for future reference, that a_2 is matched with c_1 , for otherwise a_2 is adjacent to $\{b_2, b_3, c_2, c_3\}$ and a path of length $< d$ from x' to y' is created. Similarly, c_{n-1} is matched with a_n . Thus S is as shown in Figure 10. Further, all additional edges joining C to A have positive slope (in terms of the geometric representation chosen), while those joining B to C or A have positive slope or are vertical.

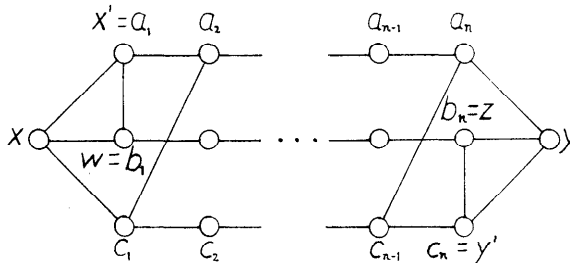


FIG. 10. Labeled representation of an ambiguous n -seine for $n \geq 4$.

At this point a certain relationship of $\{x', y'\}$ to $\{x, y\}$ has been established. A third diametral pair $\{x'', y''\}$ would bear this same relationship to *both* $\{x, y\}$ and $\{x', y'\}$, whence $\{x'', y''\} = \{w, z\}$ for (with $n \geq 4$) w and z are the only two nodes adjacent to both $\{x, y\}$ and $\{x', y'\}$. Since $\{w, z\}$ is not diametral it follows that $\{x, y\}$ and $\{x', y'\}$ are the only diametral pairs. ■

It follows from 3.1 that the number of isomorphism types of rooted n -seines that arise from a given ambiguous n -seine is at most 4. It is 1 for each of the 2-seines, 2 for the upper right 4-seine of Figure 4 (which is ambiguous and is isomorphic to the one on its left), and 4 for the ambiguous 8-seine of Figure 11. In order to use 2.5 and 2.6 in counting the total number of n -seines we must determine, for each $t \in \{1, 2, 3, 4\}$, how many types of ambiguous n -seine yield precisely t types of rooted n -seine. That is done by first counting the number of types of rooted ambiguous n -seines and then taking symmetries into account.

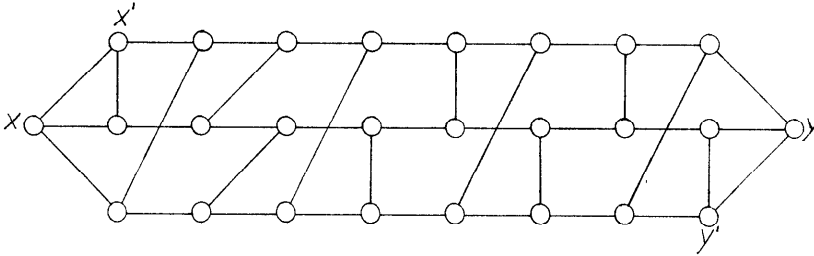


FIG. 11. An ambiguous 8-seine that yields four different types of rooted 8-seines.

For a rooted n -seine $R = (S, x, y)$ and for $1 \leq i \leq n$, let R_i denote the set of all nodes v of S such that $\delta_S(x, v) = i$; the members of R_i are said to be of *level* i . Note that $|R_i| = 3$ for each i , and each R_i is carried onto itself by each automorphism of R . For each interval I of integers in $[1, n]$, let $R_I = \bigcup_{i \in I} R_i$. An l -*block* of R , or *block of length* l , is an interval $I \subset [1, n]$ such that

- (a) $|I| = l$,
- (b) for each $p \in R_I$, at least two of the three S -neighbors of p also belong to R_I , and
- (c) I is minimal with respect to (b).

Plainly each block-length is even. The *specification* of R is the sequence $H = (h_1, \dots, h_b)$ of half-lengths of the successive blocks. Thus $\sum_{i=1}^n h_i = n/2$. For the 8-seine S of Figure 11, the specification of (S, x, y) is $(2, 1, 1)$ and of (S, x', y') is $(1, 3)$.

A rooted n -seine $R = (S, x, y)$ is *rigid* if S admits only one set of three independent paths from x to y . A *switch* is a 2-block I such that for each $i \in I$, no edge of S joins two points of R_i ; equivalently, each node in either of the two levels represented in R_I has two neighbors in the other level. A *cross* is a set of four nodes that span a circuit whose nodes alternate between two levels. The nonrigid rooted 10-seine shown below has a switch in levels 3–4 and a cross in levels 8–9. The 4-circuit in levels 6–7 does not correspond to a cross because its nodes do not alternate between two levels.

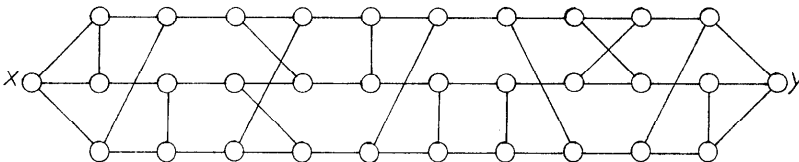


FIG. 12. A nonrigid rooted 10-seine having one switch and one cross.

Let us say that a rooted n -seine $R = (S, x, y)$ is *canonically labeled* when each node other than x or y has been assigned a unique label from the set $\bigcup_{i=1}^n \{a_i, b_i, c_i\}$ in such a way that $A = (x, a_1, \dots, a_n, y)$, $B = (x, b_1, \dots, b_n, y)$ and $C = (x, c_1, \dots, c_n, y)$ are independent paths from x to y and $\{a_1, b_1\}$, $\{c_1, a_2\}$ are edges of S .

3.2. PROPOSITION. *Suppose that R is a rooted n -seine (S, x, y) . Then*

- (a) *R admits a canonical labeling if and only if R does not have a switch in levels 1-2;*
- (b) *R is rigid if and only if R has no switches and no crosses;*
- (c) *if R is rigid then R admits a unique canonical labeling;*
- (d) *if R is rigid then the identity is the only automorphism of R .*

Proof. Assertions (a)–(c) are almost obvious and are left to the reader. For (d), let R be canonically labeled and note that $\{x, a_1, b_1\}$ is the unique 3-clique containing x , whence each automorphism of R carries the set $\{a_1, b_1\}$ onto itself. But then c_1 is carried onto itself because levels are preserved, and a_2 is carried onto itself because it is the only node in level 2 that has two neighbors in level 1. When R is rigid this implies A is carried onto A and C onto C , whence B is carried onto B and the automorphism is the identity. ■

3.3. PROPOSITION. *If h_1, \dots, h_b are positive integers whose sum is $n/2$, the number of isomorphism types of rigid rooted n -seines with specification (h_1, \dots, h_b) is $3^{b-1}2^{n-b-1}$.*

Proof. It follows from 3.2 that two rigid rooted n -seines are isomorphic if and only if the correspondence of their node-sets given by the canonical labelings is an isomorphism. Thus it remains only to determine the number of matchings M which are of the sort involved in the definition of an n -seine, have neither switch nor cross, match a_1 with b_1 and c_1 with a_2 , and generate blocks according to the specification (h_1, \dots, h_b) .

If $h_1 = 1$ the part of the n -seine corresponding to the first block is already determined. If $h_1 > 1$ there are 2 ways of choosing the two edges of M that join R_2 to R_3 , then 2 ways of choosing the single edge of M that joins R_3 to R_4, \dots , and hence a total of 2^{2h_1-2} ways of completing the first block. For each subsequent block of length $2h_i$, there are 6 ways of choosing the single edge of M that joins the first two levels of the block and then (as when $i = 1$) 2^{2h_i-1} ways of completing the block. Thus the total number of rigid rooted n -seines with specification (h_1, \dots, h_b) is

$$6^{b-1}2^{2\sum_{i=1}^b(h_i-1)} = 3^{b-1}2^{n-b-1}. \quad \blacksquare$$

3.4. PROPOSITION. *Except for the first 2-seine, each rooted ambiguous n -seine is rigid. If $n \geq 4$ and h_1, \dots, h_b are positive integers whose sum is $n/2$ there is a unique isomorphism type of rooted ambiguous n -seine with specification (h_1, \dots, h_b) .*

Proof. Consider a rooted ambiguous n -seine $R = (S, x, y)$ with $n \geq 4$, let (x', y') be as in 3.1, and note that by the proof of 3.1 R admits a canonical labeling in which $x' = a_1$ and $y' = c_n$. To show R is rigid, refer to Figure 10 and note that if there is a switch or cross then $\delta_S(x', y') < d$.

The proof of 3.4 is completed by working through the successive levels to show that the remaining edges of M are uniquely determined by the specification (h_1, \dots, h_b) in conjunction with the fact that $\delta_S(x', y') = d$. Indeed, the following properties of R can be verified by induction on i :

- a block begins at level i if and only if $\{a_i, b_i\}$ is an edge;
- a block ends at level i if and only if $\{b_i, c_i\}$ is an edge;
- for each odd i , $\{c_i, a_{i+1}\}$ is an edge;
- for each even i that does not end a block, $\{b_i, a_{i+1}\}$ and $\{c_i, b_{i+1}\}$ are edges. ■

The next result could also have been proved by a recursive procedure similar to the one used in proving 2.5.

3.5. THEOREM. *The total number of isomorphism types of rigid rooted n -seines is $10^{(n-2)/2}$; of these, $2^{(n-2)/2}$ are ambiguous.*

Proof. Let $m = n/2$ and recall that $\binom{m-1}{b-1}$ is the number of ordered partitions of m into b positive integers. By 3.3 the number of rigid rooted n -seines is

$$2^{2m-2} \sum_{b=1}^m \binom{m-1}{b-1} \left(\frac{3}{2}\right)^{b-1} = 2^{2m-2} \left(1 + \frac{3}{2}\right)^{m-1} = 10^{m-1}$$

and by 3.4 the number of rooted ambiguous n -seines is

$$\sum_{b=1}^m \binom{m-1}{b-1} = 2^{m-1}. \quad \blacksquare$$

For each finite sequence $H = (h_1, \dots, h_b)$ let H^p denote the reverse sequence (h_b, \dots, h_1) . If H is the specification of (S, x, y) then H^p is the specification of (S, y, x) . An unrooted n -seine S is called *symmetric* if it admits an automorphism interchanging the two nodes of a diametral pair. If $\{x, y\}$ is the pair in question and H is the specification of (S, x, y) then $H^p = H$. Each unambiguous n -seine S yields 1 or 2 isomorphism types of rooted n -seines according as S is or is not symmetric. However, for ambiguous n -seines the

situation is more complicated and is elucidated with the aid of some additional operations on sequences.

For each finite sequence H of positive integers let H^α denote the sequence obtained from H by doing the following:

replace each maximal segment of 1's in H by $-r$, where r is the number of 1's in the segment;

insert a 0 between each pair of consecutive entries > 1 .

For example, if

$$H = (1, 1, 1, 7, 2, 1, 3, 1, 1, 4, 5)$$

then

$$H^\alpha = (-3, 7, 0, 2, -1, 3, -2, 4, 0, 5).$$

For each finite sequence K of integers let K^β be obtained from K by the simultaneous replacement of each entry k of K as follows, where s is the number of neighbors of the entry in K —that is, $s = 0$ if there is only one entry, else $s = 1$ for the first and last entries and $s = 2$ for all intermediate entries:

if $k \leq 0$ replace k by $s - k$;

if $k \geq 2$ replace k by a segment of $k - s$ 1's.

For example, if H and H^α are as above then

$$H^{\alpha\beta} = (4, 1, 1, 1, 1, 1, 2, 3, 1, 4, 1, 1, 2, 1, 1, 1, 1),$$

$$H^{\alpha\beta\alpha} = (4, -5, 2, 0, 3, -1, 4, -2, 2, -4),$$

and

$$H^{\alpha\beta\alpha\beta} = (1, 1, 1, 7, 2, 1, 3, 1, 1, 4, 5) = H.$$

It can be proved directly, or as a corollary of 3.6 below, that $H^{\alpha\beta\alpha\beta} = H$ for every finite sequence H of positive integers.

3.6. PROPOSITION. *If S, x, y, x' and y' are as in 3.1 and H is the specification of (S, x, y) then $H^{\alpha\beta}$ is the specification of (S, x', y') . In particular, the specification of (S, x, y) starts with 1 if and only if the specification of (S, x', y') does not start with 1.*

Proof. Let (S, x, y) and (S, x', y') be canonically labeled using labels a_i , b_i and c_i in the first case and labels a'_i , b'_i and c'_i in the second case. Then the pairs of labels attached to the various nodes of S are those appearing in corresponding positions in the following two lists:

$$x'a'_1a'_2 \cdots a'_ny', x'b'_1 \cdots b'_ny', x'c'_1 \cdots c'_{n-1}c'_ny'$$

$$a_1xc_1 \cdots c_{n-1}c_n, a_1b_1 \cdots b_nc_n, a_1a_2 \cdots a_nyc_n.$$

The desired conclusion is a consequence of this correspondence in conjunction with the properties of (S, x, y) listed at the end of the proof of 3.4 and the analogous properties of (S, x', y') . ■

3.7. PROPOSITION. *If S, x, y, x' and y' are as in 3.1 and H is the specification of (S, x, y) then exactly one of the following statements is true:*

- (a) $H \neq H^p \neq H^{\alpha\beta}$, S admits no nontrivial automorphism;
- (b) $H^p = H$, S admits a unique nontrivial automorphism, and it interchanges x with y and x' with y' ;
- (c) $H^p = H^{\alpha\beta}$, S admits a unique nontrivial automorphism, and it interchanges x with y' and x' with y .

Proof. If $\{p, q\}$ and $\{u, v\}$ are diametral pairs in S and ξ and η are automorphisms of S that carry p onto u , then $\xi\eta^{-1}$ is an automorphism of the rooted n -seine (S, p, q) , whence $\xi = \eta$ by 3.4 and 3.2c. It follows from 3.6 that no automorphism of S carries x onto x' . Thus perhaps

there is an automorphism ϕ of S that carries x onto y , or
there is an automorphism ψ of S that carries x onto y' ,

but there is at most one such ϕ , at most one such ψ , and there are no other automorphisms of S except the identity. With the aid of 3.4 and 3.6 it follows that:

ϕ exists if and only if $H^p = H$, and then ϕ actually interchanges x with y and x' with y' ;

ψ exists if and only if $H^p = H^{\alpha\beta}$, and then ψ actually interchanges x with y' and x' with y ;

ϕ and ψ do not both exist, because $H^{\alpha\beta} \neq H$. ■

An unrooted n -seine S is called *skew-symmetric* if $n \geq 4$ and condition (c) is satisfied. (See Figures 13 and 14.) By 3.7, no n -seine is both symmetric and skew-symmetric.

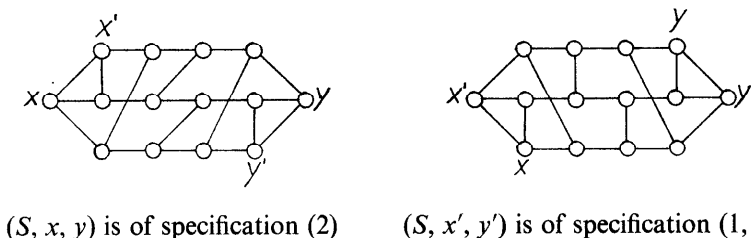


FIG. 13. Two views of a symmetric ambiguous 4-seine.

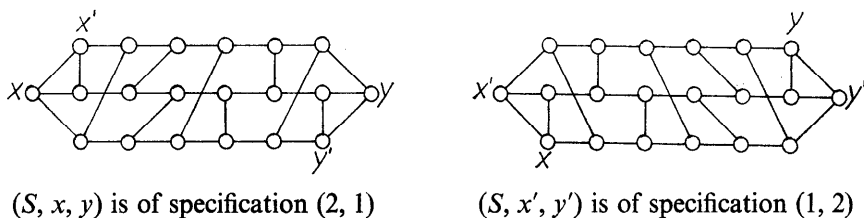


FIG. 14. Two views of a skew-symmetric ambiguous 6-seine.

The following result sharpens the second part of 3.5.

3.8. PROPOSITION. *If γ_n \langle resp. γ'_n, γ''_n \rangle is the number of rooted ambiguous n -seines for which the specification starts and ends with a 1 \langle resp. starts but does not end with a 1, neither starts nor ends with a 1 \rangle , then $\gamma_2 = \gamma_4 = \gamma'_4 = 1$, $\gamma'_2 = \gamma''_2 = \gamma'_4 = 0$, and for all $n \geq 6$,*

$$\gamma_n = \gamma'_n = \gamma''_n = 2^{(n-6)/2}.$$

Proof. The initial conditions are obvious. From the second part of 3.6, applied to ends as well as starts, it follows that $\gamma_n = \gamma''_n$ when $n \geq 4$. In view of the second part of 3.4, the following recursions are obvious: $\gamma'_{n+2} = \gamma'_n + \gamma''_n$; $\gamma_{n+4} = \gamma_n + 2\gamma'_n + \gamma''_n$. The stated conclusions follow by induction. ■

3.9. PROPOSITION. *If δ_n \langle resp. δ'_n \rangle is the number of rooted symmetric n -seines for which the specification starts and ends \langle resp. neither starts nor ends \rangle with 1, then $\delta_2 = 1$, $\delta'_2 = 0$, and for all $n \geq 4$,*

$$\delta_n = \delta'_n = 2^{\lfloor n/4 \rfloor - 1}.$$

Proof. By 3.7 and the second part of 3.6, $\delta_n = \delta'_n$ for all $n \geq 4$. In view of 3.4, $\delta_{n+4} = \delta_n + \delta'_n$. Use induction. ■

3.10. PROPOSITION. *If ϵ_n \langle resp. ϵ'_n \rangle is the number of rooted skew symmetric n -seines for which the specification starts \langle resp. does not start \rangle with 1, then $\epsilon_2 = \epsilon'_2 = 0$, and for all $n \geq 4$,*

$$\epsilon_n = \epsilon'_n = \begin{cases} 0 & \text{when } n \equiv 0 \pmod{4} \\ 2^{\lfloor n/4 \rfloor - 1} & \text{when } n \equiv 2 \pmod{4}. \end{cases}$$

Proof. By 3.6, $\epsilon_n = \epsilon'_n$ for all $n \geq 4$. To complete the proof, note that $\epsilon_2 = \epsilon'_2 = 0$, $\epsilon_4 = 0$, $\epsilon_6 = 1$, and use induction in conjunction with the

recursion $\epsilon_{n+4} = \epsilon_n + \epsilon'_n$. That $\epsilon_{n+4} = \epsilon_n + \epsilon'_n$ for all $n \geq 8$ is a consequence of the following property of the operators ρ , α and β :

If the sequence $H = (h_1, \dots, h_b)$ of positive integers satisfies the condition that $H^\alpha = H^{\alpha\beta}$ then the same condition is satisfied by the sequence $(1, h_1, \dots, h_{b-1}, h_b + 1)$; when $b \geq 2$ and $h_1 = 1$ the condition is satisfied also by the sequence $(h_2, \dots, h_{b-1}, h_b - 1)$. ■

3.11. THEOREM. Let ϕ_n , ψ_n and ω_n denote the respective numbers of ambiguous n -seines, of symmetric ambiguous n -seines, and of skew symmetric ambiguous n -seines. Then $\phi_2 = \psi_2 = \phi_4 = \psi_4 = 2$, $\omega_2 = \omega_4 = 0$, and

for all even $n \geq 4$, $\psi_n = 2^{\lfloor n/4 \rfloor - 1}$,

for all even $n \geq 4$, $\omega_n = 0$ or $2^{\lfloor n/4 \rfloor - 1}$ according as $n \equiv 0$ or $2 \pmod{4}$,

for all even $n \geq 6$, $\phi_n = 2^{(n-6)/2} + 2^{\lfloor (n-2)/4 \rfloor}$.

Proof. Use 3.9 and 3.10 for ψ_n and ω_n . With $\lambda_n = \phi_n - \psi_n - \omega_n$, it follows from 3.5 and 3.7 that

$$4\lambda_n + 2\psi_n + 2\omega_n = 2^{(n-2)/2}$$

whence

$$\phi_n = 2^{(n-6)/2} + \frac{1}{2}\psi_n + \frac{1}{2}\omega_n$$

and the stated equality follows. ■

3.12. THEOREM. Let σ_n denote the number of isomorphism types of n -seines and τ_n the number of isomorphism types of symmetric n -seines. Then for all even $n \geq 6$,

$$\sigma_n = \begin{cases} \frac{1}{2}(1 + 21\zeta_j) - 44\zeta_{j-1} + 30\zeta_{j-2} - 2^{j-1} & \text{when } n = 4j \\ 1 + \frac{1}{2}(3\zeta_{j+1} - 13\zeta_j) + 5\zeta_{j-1} - 2^{j-1} & \text{when } n = 4j + 2 \end{cases}$$

and

$$\tau_n = \begin{cases} \frac{3}{4}\zeta_{2j} - \frac{5}{2}\zeta_{2j-2} + \frac{21}{4}\zeta_j - 22\zeta_{j-1} + 15\zeta_{j-2} - 4^{j-2} & \text{when } n = 4j \\ \frac{3}{4}\zeta_{2j+1} - \frac{5}{2}\zeta_{2j} + \frac{3}{4}\zeta_{j+1} - \frac{13}{4}\zeta_j + \frac{5}{2}\zeta_{j-1} - 4^{j-1} - 2^{j-1} + \frac{1}{4} & \text{when } n = 4j + 2. \end{cases}$$

Proof. Let ϕ_n , ψ_n , ω_n and λ_n be as in 3.11, and let ρ_n denote the number of isomorphism types of reversible rooted n -seines. Each unambiguous

symmetric n -seine yields a single type of reversible rooted n -seine, and by 3.7 each ambiguous symmetric n -seine yields two such types (assuming $n \geq 4$). Hence $(\sigma_n - \psi_n) + 2\psi_n = \rho_n$ and the above equation for σ_n follows from 2.6 and 3.11. Similarly, there is a single type of rooted n -seine associated with each unambiguous symmetric n -seine, there are two such types associated with each n -seine that is unambiguous and asymmetric or ambiguous and symmetric or ambiguous and skew-symmetric, and there are four such types associated with each n -seine that is ambiguous but neither symmetric nor skew-symmetric. It follows that for even $n \geq 4$,

$$(\sigma_n - \psi_n) + 2(\tau_n - \phi_n - (\sigma_n - \psi_n)) + 2\psi_n + 2\omega_n + 4\lambda_n$$

is the number r_n of isomorphism types of rooted n -seines, whence

$$2\tau_n = r_n - 2\phi_n + \psi_n + 2\omega_n + \sigma_n.$$

The stated conclusion then follows with the aid of 2.5, 3.11 and the value of σ_n already established. ■

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